# Input to State Stabilizing Controller for Systems with Coarse Quantization

Yoav Sharon, Daniel Liberzon

***Abstract*—We consider the problem of achieving input-to-state stability (ISS) with respect to external disturbances for control systems with quantized measurements. Quantizers considered in this paper take finitely many values and have an adjustable "center" and "zoom" parameters. Both the full state feedback and the output feedback cases are considered. Similarly to previous techniques from the literature, our proposed controller switches repeatedly between "zooming out" and "zooming in". However, here we use two modes to implement the "zooming in" phases, which allows us to attenuate an unknown disturbance while using the minimal number of quantization regions. Our analysis is trajectory-based and utilizes a cascade structure of the closed-loop hybrid system. We further show that our method is robust to modeling errors using a specially adapted small-gain theorem. The main results are developed for linear systems, but we also discuss their extension to nonlinear systems under appropriate assumptions.**

***Index Terms*—Quantized systems, Stability of hybrid systems, Input-to-state stability (ISS), Disturbances**

## I. Introduction

A *quantizer* is a device that converts a real-valued signal into a piecewise constant one taking a finite set of values. In the context of feedback control systems, the real-valued signal is either the measurable output of the system or the control input. Quantization is generally a constraint related to the implementation of the control system. Digital sensors, digital controllers and data links with limited date rate are typical in many implementations of control systems, and they all induce some degree of quantization.

The study of the influence of quantization on the behavior of feedback control systems can be traced back at least to [1]. In the literature on quantization, the quantized control system is typically regarded as a perturbation of the ideal (unquantized) one. Two principal phenomena account for changes in the system's behavior caused by quantization. The first one is saturation: if the quantized signal is outside the range of the quantizer, then the quantization error is large, and the system may significantly deviate from the nominal behavior (e.g., become unstable). The second one is deterioration of performance near the target point (e.g., the equilibrium to be stabilized): as this point is approached, higher precision is required, and so the presence of quantization errors again distorts the properties of the system. These effects can be precisely characterized using the tools of system theory, specifically, Lyapunov functions and perturbation analysis; see, e.g., [2], [3], [4] for results in this direction. We refer to this line of work as the "perturbation approach". The more recent work [5], also falling into this category, is particularly relevant because it reveals the importance of input-to-state stability for characterizing the robustness of the controller to quantization errors for general nonlinear systems.

An alternative point of view which this paper follows, pioneered by Delchamps [3], is to regard the quantizer as an information-processing device, i.e., to view the quantized signal as providing a limited amount of information about the real quantity of interest (system state, control input, etc.) which is encoded using a finite alphabet. This "information approach" seems especially suitable in modern applications such as networked and embedded control systems. The main question then becomes: how much information is really needed to achieve a given control objective? In the context of stabilization of linear systems, one can explicitly calculate the minimal information transmission rate that will dominate the expansiveness of the underlying system dynamics. Results in this direction are reported in [6], [4], [7], [8], [9], [10] and in the papers cited in the next paragraph; [11], [12], [13], [14] provide extensions to nonlinear systems.

All the aforementioned works only addressed stability in the absence of external disturbances. Several papers did address the issue of external disturbances, differing mainly in the stability property they aim to achieve and in their assumptions on the external disturbance. Papers [15], [16] and [17] designed a controller which guarantees stability only for a disturbance whose magnitude is lower than some known value. In the paper [18] mean square stability in the stochastic setting is obtained by utilizing statistical information about the disturbance (a bound on its appropriate moment). The paper [19] designed a controller with which it is possible to bound the plant's state in probability. With the expense of one additional feedback bit, no further information about the disturbance is required. Note that these two latter papers use (and prove) stochastic stability notions. All of these papers followed the information approach. Deterministic stability for a completely unknown bounded disturbance was initially shown in [20]. By generalizing the perturbation approach of [4], [5], the deterministic stability property achieved in [20] is input-to-state stability (ISS) which, apart from ensuring a bounded state response to every bounded disturbance, also ensures asymptotic stability (convergence to the origin) when the disturbance converges to zero. The approach of [20] was also shown to produce $\ell_2$ stability in [14] (also, [21]).

In this paper we also address the problem of achieving ISS for deterministic systems and completely unknown disturbances. In contrast to [20], which followed the perturbation approach, our first and main contribution here is that we do this following the information approach. The main advantage of using the information approach is that it requires fewer, pos-

This work was done at the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, Illinois, USA. E-mail: ysharon@ieee.org, liberzon@illinois.edu. The work of both authors was supported by NSF ECCS-0701676 award.

sibly many fewer, quantization regions, which also translates to lower data rate. As a result, a better understanding is achieved of how much information is required for ISS disturbance attenuation. In fact, when all state variables are observed (quantized state feedback) we are able to achieve a data rate which can be arbitrarily close to the minimal data rate required for stabilization with no disturbance. We stress that following the information approach and not the perturbation approach necessitates significantly different design and analysis tools than what is described in [20].

Our second contribution is that we also consider the case where the state space is only partially measured, the situation commonly referred to as output feedback. This is a significant generalization of the approach described in [10], where only a specific observer was given and no disturbances were considered. The papers [18], [19] and [13] do formulate a system with output feedback, but it is assumed there that a state estimate is generated before the quantization is applied ([13] does not deal with disturbances). Here we generate the state estimate from the quantized measurements. We argue that this setting is much more reasonable when the quantization is due to physical or practical constraints on the sensors (as opposed to just a data rate constraint); refer to Remark 2 for more details. We emphasize that our results are novel even for the state feedback case.

Our third contribution is establishing stability under modeling errors where the system model is known only approximately, and may also vary over time. We show that under small enough modeling errors the system remains ISS in a local practical sense. We prove this robustness result using a specially adapted small-gain theorem.

The paper is organized as follows: In §II-A we define the system and the specific quantizer we use; in §II-B we define the desired stability property, an extension of the ISS property; in §III we present the proposed controller; in §IV we state and prove our main results; in §V we show that we can arbitrarily approach the minimum data-rate for the unperturbed system; finally, in §VI we show how our results can be extended to nonlinear systems. We defer to part A of the appendix the proofs of our technical lemmas. In part B of the appendix we show that the small-gain theorem applies to our modified ISS notion.

## II. Problem Statement

### A. System Definition

The linear continuous-time dynamical system we are to stabilize is as follows ($t \in \mathbb{R}_{\geq 0}$):

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + D\boldsymbol{w}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state, $\boldsymbol{x}_0 \in \mathbb{R}^{n_x}$ is an unknown initial condition, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{w} \in \mathbb{R}^{n_w}$ is an unknown disturbance, assumed to be Lebesgue-measurable and locally bounded, and $\boldsymbol{y} \in \mathbb{R}^{n_y}$ is the measured output ($n_y \leq n_x$).

While $\boldsymbol{y}$ is what the sensors measure, we assume that the information available to the controller is $\boldsymbol{z} : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}^{n_y}$, which is a sampled and quantized version of $\boldsymbol{y}$:

$$\boldsymbol{z}(kT_s) = Q(\boldsymbol{y}(kT_s); \boldsymbol{c}(kT_s), \mu(kT_s)) \tag{2}$$

where $Q$ is a quantization function and $T_s > 0$ is the time-sampling interval. The quantization parameters, $\boldsymbol{c} : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}^{n_y}$ and $\mu : \{kT_s \,|k \in \mathbb{Z}_{\geq 0}\} \to \mathbb{R}_{>0}$, are generated by the controller. For convenience we use the notation $\boldsymbol{z}_k \doteq \boldsymbol{z}(kT_s)$, and similarly for other variables, so (2) becomes $\boldsymbol{z}_k = Q(\boldsymbol{y}_k; \boldsymbol{c}_k, \mu_k)$. We refer to the special case where $C = I$, the identity matrix, as the quantized state feedback problem. We refer to the general case where $C$ is arbitrary as the quantized output feedback problem.

We consider the following (square) quantizer. Assume $N$, the number of quantization regions per observed dimension, is an odd number. The quantizer is denoted by $(Q_1, \ldots, Q_{n_y})^T = Q(\boldsymbol{y}; \boldsymbol{c}, \mu)$ where each scalar component is defined as follows (see Figure 1 for an illustration):

$$Q_i(\boldsymbol{x}; \boldsymbol{c}, \mu) \doteq c_i + 2\mu \times \begin{cases} (-N+1)/2 & x_i - c_i \leq (-N+2)\mu \\ (N-1)/2 & (N-2)\mu < x_i - c_i \\ \lceil (x_i - (c_i + \mu)) / (2\mu) \rceil & \text{otherwise.} \end{cases} \tag{3}$$

We refer to $\boldsymbol{c}$ as the *center* of the quantizer, and to $\mu$ as the *zoom factor*. Note that what will actually be transferred from the quantizer to the controller will be an index to one of the quantization regions. The controller, which either generates the values $\boldsymbol{c}$ and $\mu$ or knows the rule by which they are generated,[1] uses this information to convert the received index to the value of $Q$ as given in (3).

*Remark 1:* Our results, except for those in §V, apply to a more general family of quantizers. For an arbitrary quantizer, we denote by $\mathcal{Q}(\boldsymbol{c}, \mu)$ the (finite) set of possible values of $Q(\cdot; \boldsymbol{c}, \mu)$. A quantizer belongs to the family of quantizers to which our results apply if there exist real numbers $M > 1$ and $0 \leq H \leq N-1$ such that for all $\boldsymbol{y}$, $\boldsymbol{c}$ and $\mu$ there exists a set $\mathcal{Q}_{INT}(\boldsymbol{c}, \mu) \subsetneq \mathcal{Q}(\boldsymbol{c}, \mu)$ for which the following implications hold with an arbitrary choice of norm:

$$\begin{aligned} |\boldsymbol{y} - \boldsymbol{c}| < M\mu \qquad &\Rightarrow |Q(\boldsymbol{y}; \boldsymbol{c}, \mu) - \boldsymbol{y}| < \mu \\ |\boldsymbol{y} - \boldsymbol{c}| < (M - H)\mu \qquad &\Rightarrow Q(\boldsymbol{y}; \boldsymbol{c}, \mu) \in \mathcal{Q}_{INT}(\boldsymbol{c}, \mu) \\ Q(\boldsymbol{y}; \boldsymbol{c}, \mu) \in \mathcal{Q}_{INT}(\boldsymbol{c}, \mu) \qquad &\Rightarrow |Q(\boldsymbol{y}; \boldsymbol{c}, \mu) - \boldsymbol{y}| < \mu. \end{aligned}$$

The set $\mathcal{Q}_{INT}$ is the set of quantization regions which are bounded in the output space — no further assumption is needed to bound the quantization error if the quantizer transmits an index to a region belonging to $\mathcal{Q}_{INT}$. It is easy to see that the square quantizer above belongs to this family with $\mathcal{Q}_{INT}(\boldsymbol{y}; \boldsymbol{c}, \mu) = \{(c_1 + q_1\mu, \ldots, c_{n_y} + q_{n_y}\mu) \,|q_i \notin \{-N+1, N-1\}, \forall i\}$, $M = N$ and $H = 2$ when the $\infty$-norm is considered.

[1] The quantization parameters $\boldsymbol{c}$ and $\mu$ can be available to the sensors (or the sensor side of the communication link) depending on the source of quantization. When the quantization is due to the limited bandwidth of the communication, and there is sufficient computation capability on the sensor side of the communication link, the quantization parameters $\boldsymbol{c}$ and $\mu$ may be generated simultaneously on both sides of the communication link. When the quantization is due to the sensors, and the communication constraints between the controller and the sensors can be neglected, these quantities can be generated by the controller only and then sent to the sensors.

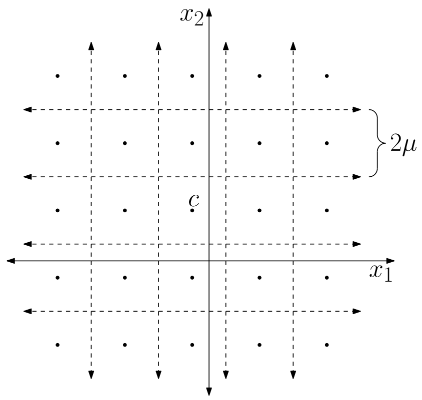


Fig. 1. Illustration of the quantizer for the two-dimensional output subspace, $N = 5$. The dashed lines define the boundaries of the quantization regions. The black dots define the quantization values.

*Remark 2:* In the literature on quantization there appear to be two different methods of positioning the partial measurement constraint (output feedback) in the feedback loop. One approach, followed by [18], [19] and [13], assumes that while not all the state variables are observed, those that are observed are measured continuously. These continuous measurements are fed into an observer that generates a state estimate. This state estimate is sent through a communication link to the controller (and thus has to be quantized). The second approach, followed by [10] and this paper, assumes that the measurements of the observed state variables are quantized, and from these quantized measurements a state estimate needs to be generated. The reason for having two approaches is the different possible sources of quantization: Both approaches can handle the case when the communication is the source of quantization; however, only the second approach can handle the case when the sensors are the source of quantization.

In this paper we use the $\infty$-norm unless otherwise specified. For vectors, $|\boldsymbol{x}| \doteq |\boldsymbol{x}|_\infty \doteq \max_i |x_i|$. For continuous-time signals, $\|\boldsymbol{w}\|_{[t_1,t_2]} \doteq \max_{t\in[t_1,t_2]} |\boldsymbol{w}(t)|_\infty$, $\|\boldsymbol{w}\| \doteq \|\boldsymbol{w}\|_{[0,\infty)}$. For discrete-time signals, $\|\boldsymbol{z}\|_{\{k_1,\dots,k_2\}} \doteq \max_{k\in\{k_1,\dots,k_2\}} |\boldsymbol{z}_k|_\infty$, $\|\boldsymbol{z}\| \doteq \|\boldsymbol{z}\|_{\{0,\dots,\infty\}}$. For matrices we use the induced norm corresponding to the specified norm ($\infty$-norm if none specified). For piecewise continuous signals we use the superscripts $+$ and $-$ to denote the right and left continuous limits, respectively: $\boldsymbol{x}_k^+ \doteq \boldsymbol{x}^+(kT_s) \doteq \lim_{t\searrow 0} \boldsymbol{x}(kT_s + t)$, $\boldsymbol{x}_k^- \doteq \boldsymbol{x}^-(kT_s) \doteq \lim_{t\nearrow 0} \boldsymbol{x}(kT_s + t)$.

### *B. Desired Stability Property*

The stability properties below are defined for a general system whose state is $\boldsymbol{\xi}$ and which is affected by an external disturbance, $\boldsymbol{w}$. In the presence of a non-vanishing disturbance, even without quantization we cannot achieve asymptotic stability. Therefore, we aim for a weaker stability property: that the system be bounded and converge to a ball around the origin whose size depends on the magnitude of the disturbance. Furthermore, when the disturbance vanishes, we expect to recover asymptotic stability. This desired behavior is encapsulated by the (global) ISS property, originally defined in [22] as follows:

$$|\boldsymbol{\xi}(t)| \le \beta(|\boldsymbol{\xi}(t_0)|, t - t_0) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right), \quad \forall t \ge t_0 \ge 0 \tag{4}$$

where $\gamma$ is a function of class $\mathcal{K}_\infty$ and $\beta$ is a function of class $\mathcal{KL}$[2].

In our system, in addition to the original state variables, $\boldsymbol{x}$, the closed-loop system contains other variables. Of these additional variables, the zoom factor in particular does not exhibit an ISS relation with respect to the disturbance. A discussion in [20, §III.B] explains why it is hard and probably impossible to have both the original state and the zoom factor exhibit an ISS relation with respect to the disturbance. Nevertheless, the value of the zoom factor at an arbitrary initial time affects the ISS relation between the disturbance and the state. Therefore, the property that we achieve, referred to as *parameterized input-to-state stability*, is defined as:

$$\begin{aligned} &|\boldsymbol{\xi}(t)| \le \beta(|\boldsymbol{\xi}(t_0)|, t - t_0; \mu(t_0)) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu(t_0)\right) \\ &\mu(t) \le \delta\left(\|\boldsymbol{\xi}\|_{[t_0,t]}; \mu(t_0)\right), \quad \forall t \ge t_0 \ge 0 \end{aligned} \tag{5}$$

where the functions $\beta(\cdot,\cdot;\cdot)$ and $\gamma(\cdot;\cdot)$ are of class $\overline{\mathcal{KL}}$ and class $\overline{\mathcal{K}}_\infty$, respectively. We say that a function $\beta : \mathbb{R}^3_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class $\overline{\mathcal{KL}}$ when, as a function of its first two arguments with the third argument fixed, it is of class $\mathcal{KL}$, and it is a continuous function of its third argument when the first two arguments are fixed. We say that a function $\gamma : \mathbb{R}^2_{\ge 0} \to \mathbb{R}_{\ge 0}$ is of class $\overline{\mathcal{K}}_\infty$ when as a function of its first argument with the second argument fixed, it is of class $\mathcal{K}_\infty$, and it is a continuous function of its second argument when the first argument is fixed. If (5) only holds locally, i.e. there exist $x_{\max} > 0$ and $w_{\max} > 0$ with which (5) holds for all $|\boldsymbol{\xi}(0)| \le x_{\max}$ and all $\|\boldsymbol{w}\| \le w_{\max}$, then we say that the system has *local parameterized input-to-state stability*.

In the case of modeling errors, even this cannot in general be achieved. Namely, we cannot achieve a global result, only a local one; furthermore, even with no external disturbance, the system is only practically stable, not asymptotically stable. The weaker result we do achieve in the case of modeling error is *local practical input-to-state stability*: There exist $\xi_{\max}$, $w_{\max}$ and $\delta_{A,\max}$ such that if $\delta_A \le \delta_{A,\max}$ where $\delta_A \in \mathbb{R}_{\ge 0}$ is a measure of the modeling errors, then

$$\begin{aligned} &|\boldsymbol{\xi}(t)| \le \beta(|\boldsymbol{\xi}(t_0)|, t - t_0) + \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + \lambda(\delta_A), \\ &\forall t \ge t_0 \ge 0 \quad \forall |\boldsymbol{\xi}(0)| < \xi_{\max} \quad \forall \|\boldsymbol{w}\|_{[0,t]} < w_{\max}. \end{aligned} \tag{6}$$

In (6) $\beta$ is a function of class $\mathcal{KL}$, and $\gamma$ and $\lambda$ are functions of class $\mathcal{K}_\infty$. This property is along the lines of the input-to-state practical stability (ISpS) [23]. The absence of the dependence on $\mu$ in (6) is due to the local nature of this stability property.

## III. Controller Design

### *A. Overview of the Controller Design*

Our controller switches between three different modes of operation. The motivation for each of these modes is given in this subsection, with a flow chart appearing in Figure 2.

[2]A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and also unbounded. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$.

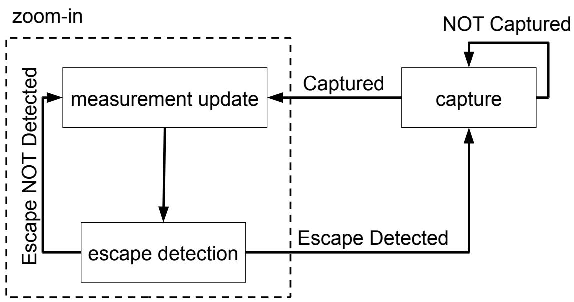


Fig. 2. Flow chart of the different modes of operations.

Our quantizer consists of quantization regions of finite size, for which the quantization error, $\boldsymbol{e}_k \doteq \boldsymbol{z}_k - \boldsymbol{y}_k$, can be bounded, and regions of infinite size, where the quantization error is unbounded. We refer to these regions as bounded and unbounded quantization regions, respectively. Only a subset of finite size of the infinite-size output space $\mathbb{R}^{n_y}$ can be covered by the bounded quantization regions. However, the size of this subset, referred to as the unsaturated region, can be adjusted dynamically by changing the parameters of the quantizer. Our controller follows the general framework that was introduced in [4], [5] to stabilize the system from an unknown initial condition using dynamic quantization. In [20] this approach was developed further to achieve disturbance attenuation. This framework consists of two main modes of operation, generally referred to as *zoom-in* and *zoom-out* modes. During the *zoom-out* mode, the unsaturated region is enlarged until the measured output is captured in this region and a state estimate with a bounded estimation error can be established. This is followed by a switch to the *zoom-in* mode. During the *zoom-in* mode, the size of the quantization regions is reduced in order to achieve convergence of the estimation error. This reduction also reduces the size of the unsaturated region, and eventually the disturbance may drive the measured output outside this region. To regain a new state estimate with a bounded estimation error, the controller switches back to the *zoom-out* mode. By switching repeatedly between these two modes, an ISS relation can be established. We use the name *capture* mode for the *zoom-out* mode.

To achieve the minimum data-rate, however, we are required to use the unbounded regions not only to detect saturation, but also to reduce the estimation error. We accomplish this dual use by dividing the *zoom-in* mode into two modes: a *measurement-update* mode and an *escape-detection* mode. After receiving $r$ successive measurements in bounded quantization regions, where $r$ is the observability index of the pair $(A, C)$, we are able to define a region in the state space which must contain the state if there were no disturbance. We enlarge this region proportionally to its current size to accommodate some disturbance. In the *measurement-update* mode we cover this containment region using both the bounded and the unbounded regions of the quantizer. This allows us to use the smallest quantization regions, leading to the fastest reduction in the estimation error. However, we cannot detect a strong disturbance in this mode. Therefore, in the *escape-detection* mode we use larger quantization regions to cover the containment region using only the bounded regions. If a strong disturbance does come in, we can detect it as the quantized output measurement will correspond to one of the unbounded regions.

### B. Preliminaries

In this section we assume that $A \equiv A_0$ is fixed and known. Extension to varying, unknown $A$ will be discussed in §IV-C. We define the sampled-time versions of $A$, $\boldsymbol{u}$ and $\boldsymbol{w}$ as ($k \in \mathbb{Z}_{\geq 0}$):

$$A_d \doteq \exp\left(T_s A_0\right), \quad \boldsymbol{x}_k \doteq \boldsymbol{x}\left(kT_s\right),$$
$$\boldsymbol{u}_k^d \doteq \int_0^{T_s} \exp\left(A_0\left(T_s - t\right)\right) B\boldsymbol{u}\left(kT_s + t\right) dt,$$
$$\boldsymbol{w}_k^d \doteq \int_0^{T_s} \exp\left(A_0\left(T_s - t\right)\right) D\boldsymbol{w}\left(kT_s + t\right) dt.$$

With these definitions we can write

$$\boldsymbol{x}_{k+1} = A_d \boldsymbol{x}_k + \boldsymbol{u}_k^d + \boldsymbol{w}_k^d. \tag{7}$$

We assume that $(A_0, B)$ is a controllable pair, so there exists a matrix $K$ such that $A_0 + BK$ is Hurwitz. By construction $A_d$ is full rank, and in general (unless $T_s$ belongs to some set of measure zero) the observability of the pair $(A_0, C)$ implies that $(A_d, C)$ is an observable pair (see [24, Proposition 6.2.11]). Thus with $r \in \mathbb{N}$, the observability index, the matrix

$$\widetilde{C} \doteq \begin{pmatrix} CA_d^{-r+1} \\ \vdots \\ CA_d^{-1} \\ C \end{pmatrix} = \begin{pmatrix} C \\ CA_d \\ \vdots \\ CA_d^{r-1} \end{pmatrix} A_d^{-r+1} \tag{8}$$

has full column rank. For state feedback systems $r = 1$ and $\tilde{C}$ is the identity matrix.

### C. Controller Architecture

Our controller consists of three elements: an observer which generates a state estimate $\hat{x}(t)$ (with the notation $\hat{x}_k \doteq \hat{x}(kT_s)$); a switching logic which updates the parameters for the quantizer and sends update commands to the observer; and a stabilizing control law which computes the control input based on the state estimate. For simplicity of presentation, we assume the stabilizing control law consists of a static nominal state feedback:

$$\boldsymbol{u}(t) = K\hat{\boldsymbol{x}}(t). \tag{9}$$

However, any control law that renders the closed-loop system ISS with respect to the disturbance and the state estimation error will work with our controller.

Given an update command from the switching logic, the observer generates an estimate of the state based on current and previous quantized measurements. We require the state estimate to be exact in the absence of measurement error and disturbance, and to be a linear function of the measurements. For concreteness, we use the following state estimate from [10] which is based on the pseudo-inverse, $\widetilde{C}^\dagger \doteq \left(\widetilde{C}^T\widetilde{C}\right)^{-1}\widetilde{C}^T$:

$$\hat{\boldsymbol{x}}_k = G\left(\boldsymbol{z}; \boldsymbol{u}^d; k\right) \doteq \widetilde{C}^\dagger \begin{bmatrix} \boldsymbol{z}_{k-r+1} + C\sum_{i=1}^{r-1} A_d^{-i}\boldsymbol{u}_{k-r+i}^d \\ \vdots \\ \boldsymbol{z}_{k-1} + CA_d^{-1}\boldsymbol{u}_{k-1}^d \\ \boldsymbol{z}_k \end{bmatrix}. \tag{10}$$

In [25] we presented additional approaches to generate a state estimate that satisfy the above requirements, and compared their properties. Note that we must have at least $r$ successive measurements to generate a state estimate. Therefore, (10) is defined only for $k \geq r-1$. In the special case of state feedback, on which we will comment further as we present our results, the state estimate is generated simply as $\hat{\boldsymbol{x}}_k = \boldsymbol{z}_k$. Between update commands the observer continuously updates the state estimate based on the nominal system dynamics:

$$\dot{\hat{\boldsymbol{x}}}(kT_s+t) = A_0\hat{\boldsymbol{x}}(kT_s+t) + B\boldsymbol{u}\left(kT_s+t\right), \qquad t \in [0, T_s)\,. \tag{11}$$

### D. Switching Logic

The switching logic keeps and updates a discrete time step variable, $k \in \mathbb{N}$, whose value corresponds to the current sampling time of the continuous system – at each sampling time, the switching logic updates $\hat{x}_k \doteq \hat{x}\left(kT_s\right)$ where $k$ is the discrete time step. At each discrete time step, the switching logic operates in one of three modes: *capture*, *measurement update* or *escape detection*. The current mode is stored in the variable $mode(k) \in \{\textbf{capture}, \textbf{update}, \textbf{detect}\}$. The switching logic also uses $p_k \in \mathbb{Z}$ and $saturated(k) \in \{\textbf{true}, \textbf{false}\}$ as auxiliary variables.

We assume the control system is activated at $k=0$ ($t=0$). We initialize $\hat{\boldsymbol{x}}_0 = \boldsymbol{0}$, $mode(0) = \textbf{capture}$, $p_0 = 0$, and $\mu_{-1} = s$, where $s$ can be any positive constant and is regarded as a design parameter. We also have the following design parameters: $\alpha \in \mathbb{R}_{>0}$, $\Omega_{\text{out}} \in \mathbb{R}$ such that $\Omega_{\text{out}} > \|A\|$, and $P \in \mathbb{Z}$ such that $P \geq r+1$. The parameter $\alpha$ corresponds to the proportional expansion of the zoom factor, $\mu$, at each sampling time. This proportional expansion prevents the state from leaving the unsaturated region when the disturbance is small relative to the current value of $\mu$. Increasing $\alpha$, subject to constraint (16) below, improves the stability to the disturbance at the expense of lowering the convergence rate. The parameter $\Omega_{\text{out}}$ corresponds to the expansion rate of the zoom factor during the zoom-out phase. The parameter $P$ corresponds to the number of sampling times between each initiation of an escape-detection sequence during the zoom-in phase. Increasing $P$ improves the convergence rate and allows for the use of fewer quantization regions. However, increasing $P$ also prolongs the time it takes to detect that the state had left the unsaturated region due to a large disturbance, and therefore the stability to disturbances is negatively affected. We also define

$$F\left(\mu;k\right) \doteq \left\|CA_d\widetilde{C}^{\dagger}\right\| \|\mu\|_{\{k-r,\ldots,k-1\}} \tag{12}$$

which in the case of state feedback reduces to $F\left(\mu;k\right) \doteq \left\|A_d\right\| \mu_{k-1}$.

At each discrete time step, $k$, the switching logic is implemented by sequentially executing the following algorithms (we use the notation $\left(z_k\right)_i$ to denote the $i$-th element of the vector $z_k$):

**Algorithm 1** preliminaries

**if** $mode(k) = \textbf{capture}$ **then**
  set $\mu_k = \Omega_{out}\mu_{k-1}$
**else if** $mode(k) = \textbf{update}$ **then**
  set

$$\mu_k = \frac{F\left(\mu;k\right) + \alpha\|\mu\|_{\{k-r-p_{k-1},\ldots,k-1-p_{k-1}\}}}{N} \tag{13}$$

**else if** $mode(k) = \textbf{detect}$ **then**
  set

$$\mu_k = \frac{F\left(\mu;k\right) + \alpha\|\mu\|_{\{k-r-p_{k-1},\ldots,k-1-p_{k-1}\}}}{N-2} \tag{14}$$

**end if**
have the observer record $\boldsymbol{z}_k = Q\left(\boldsymbol{y}(kT_s); C\hat{\boldsymbol{x}}_k, \mu_k\right)$
**if** $\exists i$ such that $\left(\boldsymbol{z}_k\right)_i = \left(C\hat{\boldsymbol{x}}_k\right)_i \pm (N-1)\mu_k$ **then**
  set $saturated(k) = \textbf{true}$
**else**
  set $saturated(k) = \textbf{false}$
**end if**
initialize $mode(k+1) = mode(k)$

**Algorithm 2** *capture* mode

**if** $mode(k) = \textbf{capture}$ **then**
  **if** $saturated(k)$ **then**
    set $p_k = 0$
  **else**
    set $p_k = p_{k-1} + 1$
    **if** $p_k = r$ **then**
      set $p_k = 0$
      have the observer update the state estimate:
        $\hat{\boldsymbol{x}}_k = G\left(\boldsymbol{z}; \boldsymbol{u}_d; k\right)$
      set $mode(k+1) = \textbf{update}$
    **end if**
  **end if**
**end if**

**Algorithm 3** *measurement update* mode

**if** $mode(k) = \textbf{update}$ **then**
  set $p_k = p_{k-1} + 1$
  have the observer update the state estimate:
    $\hat{\boldsymbol{x}}_k = G\left(\boldsymbol{z}; \boldsymbol{u}_d; k\right)$
  **if** $p_k = P - r$ **then**
    set $mode(k+1) = \textbf{detect}$
  **end if**
**end if**

**Algorithm 4** *escape detection* mode

**if** $mode(k) = \textbf{detect}$ **then**
  **if not** $saturated(k)$ **then**
    set $p_k = p_{k-1} + 1$
    have the observer update the state estimate:
      $\hat{\boldsymbol{x}}_k = G\left(\boldsymbol{z}; \boldsymbol{u}_d; k\right)$
    **if** $p_k = P$ **then**
      set $p_k = 0$, $mode(k+1) = \textbf{update}$
    **end if**
  **else**
    set $p_k = 0$, $\mu_k = s$, $mode(k+1) = \textbf{capture}$
  **end if**
**end if**

## IV. Main Results

### A. The Convergence Property

We define the following convergence property. It implies that in an infinite sequence in which the switching logic is never in the *capture* mode (a result of having no disturbance), $\lim_{k\to\infty}\mu_k = 0$. Set $\mu'$ as

$$\begin{aligned} \mu'_k &= 1, && k \in \{0,\ldots,r-1\} \\ \mu'_k &= \frac{F(\mu';k)+\alpha}{N}, && k \in \{r,\ldots,P-1\} \\ \mu'_k &= \frac{F(\mu';k)+\alpha}{N-2}, && k \in \{P,\ldots,P+r-1\}. \end{aligned} \tag{15}$$

If there exists $\sigma < 1$ for which the following holds:

$$\|\mu'\|_{\{P,\ldots,P+r-1\}} \le \sigma, \tag{16}$$

then we say that the controller has the *convergence property*.

Whether the controller has the convergence property depends on the choice of the design parameters $\alpha$ and $P$. The following Lemma (proved in the Appendix) gives a sufficient and easy to verify condition for the existence of design parameters with which the controller will have the convergence property.

*Lemma 1:* If the following condition holds:

$$\sigma_{pi} \doteq \frac{1}{N}\left\|CA_d\widetilde{C}^{\dagger}\right\| < 1 \tag{17}$$

then it is possible to choose $P$ and $\alpha$ such that the controller will possess the convergence property.

In the state feedback case we do not need an observer as the updates of the state estimate become simply $\hat{\boldsymbol{x}}_k = G(\boldsymbol{z},\boldsymbol{u}_d,k) = \boldsymbol{z}_k$. In this case (17) becomes $\|A_d\|/N < 1$.

### B. Results for When the System Model Is Known

The state estimation error is defined as

$$\tilde{\boldsymbol{x}}(t) \doteq \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t). \tag{18}$$

In the simpler case where $A \equiv A_0$, the evolution of the state estimation error is independent of the state. This property is critical in proving the following proposition, which is the main technical step for deriving the desired stability results.

*Proposition 2:* If we implement the controller with the above algorithm and that controller has the convergence property, then the state estimation error of the closed-loop satisfies the parameterized-ISS property, (5), with $\boldsymbol{\xi} = \tilde{\boldsymbol{x}}$.

The aggregate state of the system is $\left(\boldsymbol{x}^T, \hat{\boldsymbol{x}}^T\right)^T$. However, due to the analysis that follows it will be easier to state the results for the state $\left(\boldsymbol{x}^T, \tilde{\boldsymbol{x}}^T\right)^T$, which relates to the former by a simple transformation of coordinates. Our first stability result is the following:

*Theorem 1:* If we implement the controller with the above algorithm and that controller has the convergence property, then the aggregate state of the closed-loop system satisfies the parameterized-ISS property, (5), with $\boldsymbol{\xi} = \left(\boldsymbol{x}^T, \tilde{\boldsymbol{x}}^T\right)^T$.

In Theorem 1 the second inequality of (5) can actually be written as $\mu(t) \le \delta\left(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}; \mu(t_0)\right)$. We also remark that when considering $t_0 = 0$, where $\tilde{\boldsymbol{x}}(0) = \boldsymbol{x}(0)$ and $\mu(0)$ is a design parameter, Theorem 1 gives us the existence of functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, such that

$$|\boldsymbol{x}(t)| \le \beta(|\boldsymbol{x}(0)|, t) + \gamma\left(\|\boldsymbol{w}\|_{[0,t]}\right), \qquad \forall t \ge 0. \tag{19}$$

Following is an outline of the proof. We divide the trajectory of the estimation error into three repeating phases. In the first phase the system is in capture mode, and we show using Lemma 6 that in finite time the estimation error will be captured and the system will switch to the second phase. In both the second and third phases the system switches repeatedly between the measurement update and escape detections modes. However, in the second phase the zoom factor, $\mu$, is sufficiently large compared to the disturbance so that the system is guaranteed, by Lemma 4, not to switch to the capture mode. In the third phase the zoom factor is small compared to the disturbance and this guarantee is lost, but by Lemma 5 we can still bound the trajectory during that phase. In Lemma 3 we prove that the zoom factor keeps contracting during these last two phases. Lemma 7 addresses the case of small disturbance when the trajectory goes into the second phase after only $r$ sampling times from when the system last switched to the capture mode. That lemma bounds the trajectory during both the first and second phases, and states that this bound goes to zero as the disturbance and the initial condition go to zero. The three phases discussed here are defined in the proof below using $k_1$, $k_2$ and $k_3$ where $k_1$ is the beginning of the second phase, $k_2$ the beginning of the third phase, and $k_3$ the beginning of a new first phase.

An illustrative simulation of the proposed controller is given in Figure 3.

The proofs of Proposition 2 and Theorem 1 will follow the statements of the technical lemmas below. The proofs of the technical lemmas are deferred to appendix A.

*Lemma 3:* Assume that for some time step $k'$ we have $mode(k'+1) = update$ and $p_{k'} = 0$ (i.e. a *measurement update* sequence starts at $k'+1$). If $\forall k \in \{k'+1,\ldots,k'+P+1\}$, $mode(k) \ne capture$ (i.e. by time step $k'+P$ the controller has not switched to the *capture* mode) then $\|\mu\|_{\{k'-r+1+P,\ldots,k'+P\}} \le \sigma\|\mu\|_{\{k'-r+1,\ldots,k'\}}$.

*Lemma 4:* There exist constants $\zeta_D > 0$ and $\zeta_\mu > 0$ with the following properties: If for some time step $k'$ we have $mode(k'+1) = update$ and $p_{k'} = 0$, and the input is such that

$$\|\mu\|_{\{k'-r+1,\ldots,k'\}} > \frac{1}{\alpha}\zeta_D\left\|\boldsymbol{w}^d\right\|_{\{k'-r+1,k'+P-2\}}, \tag{20}$$

then $mode(m) = update\ \forall m \in \{k'+2,\ldots,k'+P-r\}$, $mode(m) = detect\ \forall m \in \{k'+P-r+1,\ldots,k'+P\}$, $mode(k'+P+1) = update$, and

$$\|\tilde{\boldsymbol{x}}\|_{\{k',\ldots,k'+P-1\}} \le \zeta_\mu\|\mu\|_{\{k'-r+1,\ldots,k'\}}. \tag{21}$$

*Lemma 5:* Assume that for some time step $k'$ we have $mode(k'+1) = update$ and $p_{k'} = 0$. Let $k_3 = \min\{k'+P, \min\{k\,|mode(k+1) = capture, k > k'\}\}$. There exists a constant $\zeta_w > 0$ such that if the disturbance does not satisfy (20), then

$$\|\tilde{\boldsymbol{x}}\|_{\{k',\ldots,k_3-1\}} \le \zeta_w\left\|\boldsymbol{w}^d\right\|_{\{k'-r+1,\ldots,k'+P-2\}}.$$

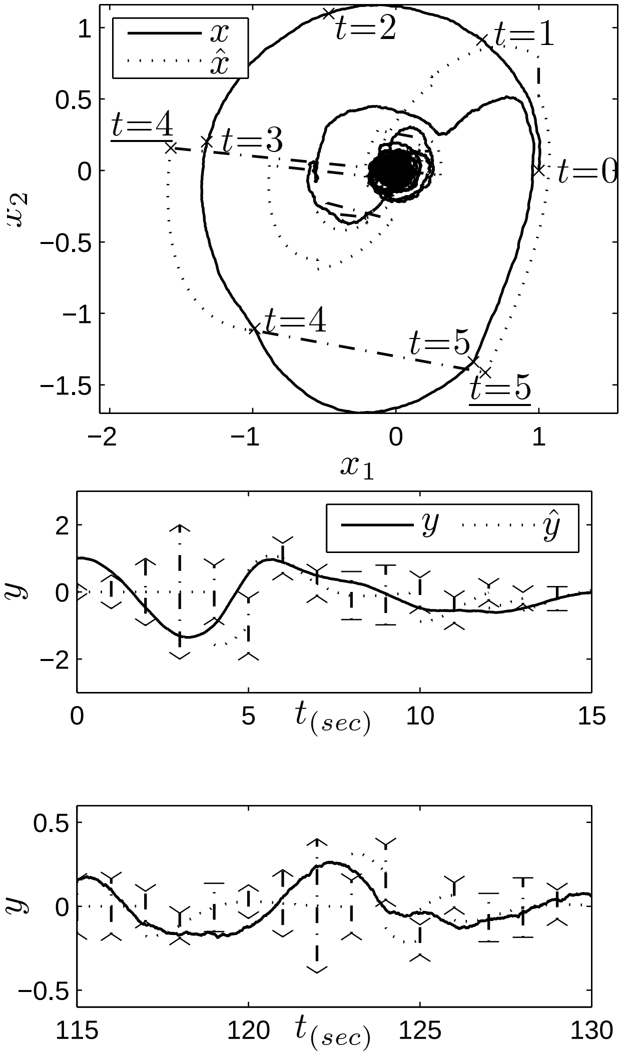


Fig. 3. Simulation of the proposed controller. Simulated here is a two-dimensional dynamical system: $\dot{\boldsymbol{x}}(t) = [0.1, -1; 1, 0.1]\, \boldsymbol{x}(t) + [0; 1]\, \boldsymbol{u}(t) + [1, 0; 0, 1]\, \boldsymbol{w}(t)$, where only the first dimension is observed, $\boldsymbol{y}(t) = [1, 0]\, \boldsymbol{x}(t)$, through a quantizer with $N = 3$. The solid line in the left plot is the trajectory of the system (starting at $\boldsymbol{x}(0) = [1; 0]$). The dotted line in that plot is the state estimate. The dash-dotted lines represent the jumps in the state estimate after a new measurement is received. The locations of the trajectory and the state estimate at the first few sampling times are marked by $\times$. The underlined time indications correspond to the state estimate. The two plots on the right show time segments of the measured output ($T_s = 1s$). The solid line is the unquantized output ($\boldsymbol{y}$) of the system and the dotted line is its estimate. The vertical dash-dotted lines depict the single bounded quantization region. The controller is in the *capture* mode where these vertical lines are bounded by arrows facing outward, in the *update* mode where the arrows are facing inward, and in the *detect* mode where the vertical lines are bounded by small horizontal lines. Looking at both the left plot and the top right plot, one can observe the initial transient of the system: At $t = 3$ a sufficient number (two) of unsaturated measurements were collected and the controller switches to the *update* mode; this causes the state estimate to jump at $t = 4$ from the origin to $\sim [-1.6; 0.2]$; and at $t = 5$ the state estimate jumps even closer to the true state. Looking at the bottom right plot, one can observe the steady-state behavior of the simulation, where an escape of the trajectory due to a disturbance is detected at $t = 119s$, and then the trajectory is recaptured at $t = 122s$. The design parameters were: $P = 6$, $\mu(0) = 0.25$, $\Omega_{out} = 2$, $\alpha = 0.02$, $s = 0.05$, $K = [0.6, -1.5]$. The disturbance followed the zero-mean normal distribution with standard deviation of $0.2$.

*Lemma 6:* There exist functions $\tilde{\delta}_1 : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and $T_1^* : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$, each nondecreasing in $\nu$ when $\rho$ is fixed, and constants $\zeta_C > 0$ and $\zeta_b > 0$, with the following properties: For any time step $k_0$ such that $mode(k_0 + 1) = capture$ there exists $k_1 > k_0$ such that $k_1 < k_0 + T_1^*\left(|\tilde{x}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$, $mode(k_1 + 1) = update$, $p_{k_1} = 0$, $\|\tilde{\boldsymbol{x}}\|_{\{k_0,\ldots,k_1\}} \leq \tilde{\delta}_1\left(|\tilde{x}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$ and $\|\mu\|_{k_1-r+1,\ldots,k_1} \leq \mu_{k_0}\Omega_{out}^{T_1^*(\nu;\rho)}$; the functions $\tilde{\delta}_1$ and $T_1^*$ satisfy $\tilde{\delta}_1(\nu;\rho) \leq \rho\zeta_b\Omega_{out}^{T_1^*(\nu;\rho)}$ $\forall \nu, \rho$.

*Lemma 7:* There exist a constant $\zeta_s > 0$, a class $\mathcal{K}$ function $\varepsilon$, and functions $\tilde{\delta}_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and $T_2^* : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ with the following properties: For any time step $k_0$ such that $|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\| \leq \varepsilon(\mu_{k_0})$, where $\zeta_C$ was defined in Lemma 6, then $k^* \doteq k_0 + T_2^*\left(|\tilde{x}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$ satisfies $\|\tilde{\boldsymbol{x}}\|_{\{k_0,\ldots,k^*\}} \leq \tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$, $\|\mu\|_{k^*-r+1,\ldots,k^*} \leq \mu_{k_0}\zeta_s\sigma^{T_2^*/P}$ where $\sigma$ was defined as part of the convergence property, $mode(k^* + 1) = update$ and $p_{k^*} = 0$; when $\rho$ is fixed the function $\tilde{\delta}_2(\cdot;\rho)$ is of class $\mathcal{K}_\infty$; the functions $\tilde{\delta}_2$ and $T_2^*$ satisfy $\tilde{\delta}_2(\nu;\rho) \leq \rho\zeta_s\sigma^{T_2^*(\nu;\rho)/P} / \|C\|$ $\forall \nu, \rho$.

*Proof of Proposition 2:* Assume that $t_0 = k_0 T_s$ for some $k_0$. We say that an arbitrary sampling time $k_2$ has the *SS properties* if $mode(k_2 + 1) = update$, $p_{k_2} = 0$ and (20) does not hold with $k' = k_2$. The proof proceeds in four steps: in the first step we derive a bound on the trajectory from $k_0$ to $k_2$; in the second step we derive a bound on the trajectory from $k_2$ to infinity; in the third step we combine these two bounds and derive the ISS bound on the estimation error; in the fourth step we derive the bound on the zoom factor.

**Step 1.** Assume first that $mode(k_0) = capture$. Let $k_2$ be the first time step after $k_0$ with the SS properties. If such a time step does not exist, define $k_2 \doteq \infty$. By Lemma 6 there exists $k_1 \leq k_0 + T_1^*\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$ such that $\|\tilde{\boldsymbol{x}}\|_{k_0,\ldots,k_1} \leq \tilde{\delta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$. With Lemmas 6, 3 and 4, we also have that if $k_2 > k_1$ then $|\tilde{\boldsymbol{x}}_k| \leq \zeta_\mu \mu_{k_0} \Omega_{out}^{T_1^*} \sigma^{\left\lfloor \frac{k-k_1}{P} \right\rfloor} \leq \zeta_\mu \mu_{k_0} \Omega_{out}^{T_1^*} \sigma^{\left\lfloor \frac{k-T_1^*}{P} \right\rfloor}$ $\forall k \in \{k_1, \ldots, k_2\}$. As Lemma 6 also states that $\tilde{\delta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right) \leq \mu_{k_0}\zeta_b\Omega_{out}^{T_1^*}$, we can derive $|\tilde{\boldsymbol{x}}_k| \leq \tilde{\beta}_c\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ $\forall k \in \{k_0, \ldots, k_2\}$ where

$$\tilde{\beta}_c(\nu, k; \rho) \doteq \min\left\{\tilde{\delta}_1(\nu;\rho), \rho\left(\frac{\Omega_{out}}{\sigma^{1/P}}\right)^{T_1^*(\nu;\rho)} \sigma^{\frac{k}{P}-1} \max\{\zeta_\mu, \zeta_b\}\right\}. \tag{22}$$

If $mode(k_0) \neq capture$ then there is a time step $k_2'$, $k_0 - P < k_2' \leq k_0$, such that $mode(k_2' + 1) = update$ and $p_{k_2'} = 0$. If in addition (20) does not hold with $k' = k_2'$, then we define $k_2 = k_2'$, and thus we have, vacuously, $|\tilde{\boldsymbol{x}}_k| \leq 0$ $\forall k \in \{k_0, \ldots, k_2\}$. If (20) does hold with $k' = k_2'$, then with $k_2$ defined as the first time step after $k_0$ with the SS properties, we can write: $|\tilde{\boldsymbol{x}}_k| \leq \zeta_\mu \mu_{k_0} \sigma^{\left\lceil \frac{k-k_0}{P} \right\rceil}$ $\forall k \in \{k_0, \ldots, k_2\}$. Taking into consideration that $mode(k_0) = capture$ only if $\mu_{k_0} \geq s$, we get $|\tilde{\boldsymbol{x}}_k| \leq \tilde{\beta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ $\forall k \in$

$\{k_0, \dots, k_2\}$ where

$$\tilde{\beta}_1(\nu, k; \rho) \doteq \begin{cases} \max\left\{\tilde{\beta}_c(\nu, k; \rho), \zeta_\mu \rho \sigma \left\lceil \frac{k-k_0}{P} \right\rceil\right\} & \rho \geq s \\ \zeta_\mu \rho \sigma \left\lceil \frac{k-k_0}{P} \right\rceil & \rho < s. \end{cases} \tag{23}$$

Assume now that $|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\| \leq \varepsilon(\mu_{k_0})$ where $\varepsilon(\cdot)$ comes from Lemma 7 and set $k_2$ to be the first time step after $k_0$ with the SS properties. Then there exists $k^* = k_0 + T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$ such that $\|\tilde{\boldsymbol{x}}\|_{k_0,\dots,k^*} \leq \tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$. With Lemmas 7, 3 and 4, we also have that if $k_2 > k_1$ then $|\tilde{\boldsymbol{x}}_k| \leq \zeta_\mu \mu_{k_0} \zeta_s \sigma^{T_2^*/P} \sigma^{\left\lfloor \frac{k-T_2^*}{P} \right\rfloor}$ $\forall k \in \{k^*, \dots, k_2\}$. As Lemma 7 also gives us that $\tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right) \leq \mu_{k_0} \zeta_s \sigma^{T_2^*/P} / \|C\|$, we derive $\forall k \in \{k_0, \dots, k_2\}$: $|\tilde{\boldsymbol{x}}_k| \leq \tilde{\beta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ where

$$\tilde{\beta}_2(\nu, k; \rho) \doteq \min\left\{\tilde{\delta}_2(\nu; \rho), \rho \zeta_s \sigma^{\lfloor k/P \rfloor} \max\{\zeta_\mu, 1/\|C\|\}\right\}. \tag{24}$$

For fixed $\nu$ and $\rho$, both $\lim_{k\to\infty} \tilde{\beta}_1(\nu, k; \rho) = 0$ and $\lim_{k\to\infty} \tilde{\beta}_2(\nu, k; \rho) = 0$. Also, for fixed $k$ and $\rho$, both $\tilde{\beta}_1(\nu, k; \rho)$ and $\tilde{\beta}_2(\nu, k; \rho)$ are continuous and nondecreasing with respect to $\nu$. However, only $\tilde{\beta}_2$ satisfies $\tilde{\beta}_2(0, k; \rho) = 0$ $\forall k, \rho$, and $\tilde{\beta}_2$ is a valid bound on the trajectory only when $|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\| \leq \varepsilon(\mu_{k_0})$. Nevertheless, it is possible to construct a class $\overline{\mathcal{KL}}$ function, $\hat{\beta}(\nu, k; \rho)$, such that $\hat{\beta}(\nu, k; \rho) \geq \tilde{\beta}_2(\nu, k; \rho)$ when $\nu \leq \varepsilon(\rho)$ and $\hat{\beta}(\nu, k; \rho) \geq \tilde{\beta}_1(\nu, k; \rho)$ otherwise. With $\hat{\beta}(\nu, k; \rho)$ we can write $|\tilde{\boldsymbol{x}}_k| \leq \hat{\beta}\left(|\tilde{\boldsymbol{x}}_{k_0}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_0; \mu_{k_0}\right)$ $\forall k \in \{k_0, \dots, k_2\}$.

Note that all the functions mentioned above are continuous in $\nu$ and $\rho$, $\forall \nu \in \mathbb{R}_{\geq 0}$ and $\forall \rho \in \mathbb{R}_{>0}$. They are not, however, all continuous (or even defined) at $\rho = 0$ since $\lim_{\rho \searrow 0} T_1^*(\nu; \rho) = \infty$ for every $\nu > 0$. Nevertheless, $\hat{\beta}(\nu, k; \rho)$ is continuous at $\rho = 0$. This is due to $\varepsilon$ being of class $\mathcal{K}$, which implies that for sufficiently small $\rho$, $\hat{\beta}(\nu, k; \rho) = \tilde{\beta}_1(\nu, k; \rho) = \zeta_\mu \rho \sigma \lceil (k - k_0)/P \rceil$.

**Step 2.** Let $k_3$ be the first time step after $k_2$ such that $mode(k_3) = detect$ and $mode(k_3 + 1) = capture$. Set $k_3 = \infty$ if such a time step does not exist. Lemma 5 gives us that $\|\tilde{\boldsymbol{x}}\|_{\{k_2,\dots,k_3\}} \leq \zeta_w \left\|\boldsymbol{w}^d\right\|$. Let $k_4$ be the first time step after $k_3$ such that $mode(k_4 + 1) = update$, $p_{k_4} = 0$ and (20) does not hold with $k' = k_4$. Replacing $k_0$ with $k_3$ in the previous step, we can write

$$\begin{aligned} \|\tilde{\boldsymbol{x}}\|_{\{k_3,\dots,k_4\}} &\leq \hat{\beta}\left(|\boldsymbol{x}_{k_3}| + \zeta_C \left\|\boldsymbol{w}^d\right\|, k - k_3; \mu_{k_3}\right) \\ &\leq \hat{\beta}\left((\zeta_w + \zeta_C)\left\|\boldsymbol{w}^d\right\|, 0; s\right) \doteq \tilde{\gamma}\left(\left\|\boldsymbol{w}^d\right\|\right). \end{aligned}$$

Since $k_4$ also satisfies the SS properties as does $k_2$, we can repeat these arguments for future time steps and arrive at $\|\tilde{\boldsymbol{x}}\|_{\{k_2,\dots,\infty\}} \leq \hat{\gamma}\left(\left\|\boldsymbol{w}^d\right\|\right)$, where $\hat{\gamma}(\nu) \doteq \max\{\zeta_w \nu, \tilde{\gamma}(\nu)\}$. Note that $\hat{\gamma}(\cdot)$ is of class $\mathcal{K}_\infty$.

**Step 3.** Combining the last two steps, we can derive the first condition for the parametrized ISS property at the discrete times: for all $k \in \{0, \dots, \infty\}$,

$$|\tilde{\boldsymbol{x}}_k| \leq \beta_e(|\tilde{\boldsymbol{x}}_{k_0}|, k; \mu_{k_0}) + \gamma_e\left(\left\|\boldsymbol{w}^d\right\|; \mu_{k_0}\right)$$

where $\beta_e(\nu, k; \mu) \doteq \hat{\beta}(2\nu, k; \mu)$ and $\gamma_e(\nu; \mu) \doteq \hat{\beta}(2\zeta_C \nu, 0; \mu) + \hat{\gamma}(\nu)$. Note that indeed $\beta_e$ and $\gamma_e$ are of class $\overline{\mathcal{KL}}$ and $\overline{\mathcal{K}}_\infty$, respectively. The extension from the discrete analysis to continuous time, with the estimation error defined as $\tilde{\boldsymbol{x}}(t) \doteq \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$ for every $t \geq t_0$, can be proved along the lines of [26, Theorem 6]. This proves the first line of (5).

**Step 4.** To construct the bound on $\mu$ we consider the three phases of the trajectory: initial *capture* sequence, *zoom-in* sequences and subsequent *capture* sequences. If $mode(k_0) = capture$ we start with $\mu_{k_0}$ and we grow the zoom factor until for $r$ successive time steps we have $(N-2)\mu_k > |\tilde{\boldsymbol{y}}_k|$. Thus at the initial *capture* sequence we have

$$\|\mu\| \leq \Omega_{out}^r \max\{\mu_{k_0}, \|C\| \|\tilde{\boldsymbol{x}}\| / (N-2)\}. \tag{25}$$

At a *zoom-in* sequence we may initially enlarge $\mu$ by a factor of $\|\mu'\|$ with $\mu'$ defined according to (15). However, after this possible initial enlargement, $\mu$ is decreased by a factor of $\sigma$ every $P$ steps. At subsequent *capture* sequences we start with $\mu_k = s$ and enlarge it again until for $r$ successive time steps we have $(N-2)\mu_k > |\tilde{\boldsymbol{y}}_k|$. Therefore, we can set $\delta$ from (5) as

$$\delta(\nu, \mu_0) \doteq \|\mu'\| \Omega_{out}^r \max\{\mu_0, s, \|C\| \|\tilde{\boldsymbol{x}}\| / (N-2)\}.$$

■

*Proof of Theorem 1:* With $A + BK$ being Hurwitz, the stabilizing control law, $u = K\hat{x}$, renders the closed-loop system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + D\boldsymbol{w} = (A + BK)\boldsymbol{x} + BK\tilde{\boldsymbol{x}} + D\boldsymbol{w} \tag{26}$$

ISS with respect to the disturbance and the estimation error. Combining this ISS property with the parameterized-ISS property proved in Proposition 2, and applying a cascade argument similar to what was used to prove [22, Proposition 7.2], we can conclude that the closed-loop system is parameterized-ISS with respect to the disturbance. ■

### *C. Modeling Errors*

We represent modeling errors as $A(t) = A_0 + \Delta A(t)$ with only $A_0$ known and $\Delta A(t) \not\equiv 0$. It is assumed, though, that $\|\Delta A(t)\| \leq \delta_A$ for some $\delta_A \in \mathbb{R}_{\geq 0}$ and $\forall t \geq 0$. To deal with such modeling errors the only change needed in the design is in the stabilizing control law, where $K$ will be chosen such there exist two positive definite symmetric matrices, $P$ and $Q$, for which the following holds:

$$\begin{aligned} P(A_0 + \Delta A + BK) + (A_0 + \Delta A + BK)^T P + Q &< 0, \\ \forall \|\Delta A(t)\| &< \delta_A. \end{aligned} \tag{27}$$

It is well-known and easy to show using a Lyapunov argument that if (27) holds then the system (26) has the ISS stability property with respect to the estimation error and disturbance:

$$\begin{aligned} |\boldsymbol{x}(t)| \leq& \beta_x(\boldsymbol{x}(t_0), t - t_0) + \gamma_{x,e}\left(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}\right) + \\ & \gamma_{x,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right), \quad \forall t > t_0 > 0 \end{aligned} \tag{28}$$

where $\beta_x$ is of class $\mathcal{KL}$ and $\gamma_{x,w}$ and $\gamma_{w,x}$ are of class $\mathcal{K}_\infty$. Such a stabilizing gain matrix $K$ can be found by using linear matrix inequality (LMI) techniques [27, §7.2].

With this stabilizing control law, we derive our second stability result:

*Theorem 2:* Assume the controller has the convergence property and the stabilizing control law is chosen so that (27) holds for some $\delta_A > 0$. Then the aggregate state of the closed-loop system satisfies the local practical ISS property (6) with $\boldsymbol{\xi} = \left(\boldsymbol{x}^T, \tilde{\boldsymbol{x}}^T\right)^T$ for some $\delta_{A,\max} > 0$, $x_{max} > 0$ and $w_{max} > 0$.

*Proof (sketch):* The dynamics of the estimation error between sampling times is now

$$\dot{\tilde{\boldsymbol{x}}} = A\tilde{\boldsymbol{x}} - \Delta A\boldsymbol{x} - D\boldsymbol{w}. \tag{29}$$

Therefore its evolution is no longer independent of the state of the system. The proposed controller in this case will render the estimation error parameterized-ISS with respect to both the disturbance and the system's state:

$$\begin{aligned} |\tilde{\boldsymbol{x}}(t)| \le& \beta_e\left(\tilde{\boldsymbol{x}}(t_0), t-t_0; \mu(t_0)\right) + \\ & \gamma_{e,x}\left(\delta_A \|\boldsymbol{x}\|_{[t_0,t]}; \mu(t_0)\right) + \gamma_{e,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu(t_0)\right) \\ \mu(t) \le& \gamma_\mu\left(\|\tilde{\boldsymbol{x}}\|_{[t_0,t]}, \mu(t_0)\right), \qquad \forall t \ge t_0 \ge 0. \end{aligned}$$

Due to the interleaved dependency of $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ on each other we can no longer apply the cascade theorem. However, since $\boldsymbol{x}_1$ which follows (1) is continuous, we can now apply a variation of the small-gain theorem, Theorem 4 which is given in Appendix B, and arrive at the result stated in the theorem. ■

Note that for every fixed $\mu$, $\gamma_{e,x}(r, \mu)$ grows faster than any linear function of $r$ both at $r = 0$ and at $r = \infty$. These super-linear gains are not an artifact of our design. In [28] it was shown, using techniques from information theory, that it is impossible to achieve ISS with linear gain for any linear system with finite data rate feedback.

## V. Approaching the Minimal Data Rate

Several papers ([8],[15],[16],[18],[19]) present the same lower bound on the data rate necessary to stabilize a given system. This bound, in terms of the bit-rate ($R$) to be transmitted, is

$$R > R_{min} \doteq \frac{\sum_{|\eta_j|\ge 1} \log_2 |\eta_j|}{T_s} \tag{30}$$

where the $\eta_j$'s are the eigenvalues of the discrete open-loop matrix $\Phi \doteq \exp(AT_s)$. Note that (30) was derived as a necessary bound for asymptotic stability in the disturbance-free case. Therefore it is necessary for achieving disturbance rejection in the ISS sense, which reduces to asymptotic stability when the disturbance is zero. The following discussion shows that any data rate that satisfies (30) is sufficient for achieving ISS using our approach.

The main steps for achieving the minimum data rate are: (1) using a different $N$ at each sampling time; (2) selecting $P$ large enough, so that the effect of the reduced resolution during the escape detection mode compared to the measurement update mode becomes negligible; (3) applying the quantization separately for each unstable mode of the system.

From Lemma 1 we have that one can choose $N$ to be the smallest integer such that $N > \|A_d\|$. Note that throughout our algorithm and proofs there is no requirement that $N$ be the same at every sampling time, as long as the convergence property is satisfied. With a different $N$ at every sampling time, denoted by $N_k$, and restricting to the state feedback case, Lemma 1 can be rephrased with the following condition replacing (17): There exists $P'$ such that for all $k$, $\prod_{l=k}^{k+P'} \|A_d\| / N_l < 1$. We can therefore choose any $\tilde{N} > \|A_d\|$, where $\tilde{N}$ is the geometric average of the $N_k$'s, and still be able to satisfy the convergence property.

For unstable scalar systems where $A = a > 0$, $\|A_d\| = \exp(aT_s) = \eta_1$, and we can then choose any average bit rate $R = 1/T_s \log_2 \tilde{N} > 1/T_s \log_2 |\eta_j|$. For multidimensional systems, when $A$ is diagonalizable with real eigenvalues, we can apply a one-dimensional quantizer on each unstable mode of the system with a number of quantization regions corresponding to the growth rate of that mode. For pairs of conjugate complex eigenvalues, $\eta_j$ and $\eta_{j+1} = \overline{\eta_j}$, we can apply a rotating two-dimensional square quantizer whose rate of rotation is $\angle\eta_j$ and its number of quantization regions per dimension corresponds to a growth rate of $|\eta_j|$. This, as well as extension to non-diagonalizable systems, is explained in details in [16].

## VI. Extension to Nonlinear Systems

The crucial properties of linear systems which are used in the proof of Theorem 1 are (a) that the continuous, unquantized, closed-loop system is ISS with respect to the estimation error and the disturbance, and (b) that the update law for the estimated state between the sampling times (11) is such that the estimation error grows between these sampling times according to

$$\begin{aligned} \lim_{t \nearrow T_s} \|\tilde{\boldsymbol{x}}(kT_s + t)\| \le& \lambda_e \|\tilde{\boldsymbol{x}}(kT_s)\| + \lambda_w \|\boldsymbol{w}\|_{[kT_s,(k+1)T_s]} + \\ & \lambda_x \|\boldsymbol{x}\|_{[kT_s,(k+1)T_s]} \end{aligned} \tag{31}$$

where $\lambda_e$, $\lambda_w$ and $\lambda_x$ are known constants. For linear systems these constants are $\lambda_e = \|A_d\|$, $\lambda_w = \left\|\int_0^{T_s} \exp(A_0(T_s - t)) D dt\right\|$ and $\lambda_x = \left\|\int_0^{T_s} \exp(A_0(T_s - t)) dt\right\| \delta_A$, which follows easily from (7). If (31) holds globally, $\lambda_x = 0$ (as in the case where the exact system model is known), and the number of quantization regions allows the controller to satisfy the convergence property, then the aggregate state of quantized system satisfies the parameterized ISS property.

Neither property is unique to linear systems and both can also be formulated for nonlinear systems. This leads to a better conceptualization of our results. Consider a nonlinear system

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)) \tag{32}$$

with $\boldsymbol{y}(t) = \boldsymbol{x}(t)$ (state feedback). State feedback control laws that render unquantized systems ISS with respect to either external disturbances or measurement errors have been proposed for certain nonlinear systems; see for example the discussions in [5], [14] and the references therein. Designing state feedback control laws that render unquantized systems ISS with respect to *both* external disturbances and measurement errors is still considered an open problem. The two closest results, for systems in strict feedback form, appear in [29, §6.2.2] and [30].

Assume that (32) satisfies the Lipschitz property:

$$|f(\boldsymbol{x},\boldsymbol{u},\boldsymbol{w}) - f(\hat{\boldsymbol{x}},\boldsymbol{u},0)| \le L_x|\boldsymbol{x}-\hat{\boldsymbol{x}}| + L_w|\boldsymbol{w}|, \quad \forall|\boldsymbol{x}| < l_x, \quad \forall|\hat{\boldsymbol{x}}| < l_x, \quad \forall|\boldsymbol{w}| < l_w \tag{33}$$

for some $l_x > 0$, $l_w > 0$, $L_x > 0$ and $L_w > 0$. When the Lipschitz property holds globally, as in linear systems, $l_x = l_w = \infty$. Assuming the exact system model is known, if we update our state estimate between sampling times according to $\dot{\hat{\boldsymbol{x}}} = f(\hat{\boldsymbol{x}},\boldsymbol{u},0)$, then (31) holds with

$$\lambda_e \doteq e^{T_s L_x}, \qquad \lambda_w \doteq \int_0^{T_s} e^{(T_s-\tau)L_x} d\tau L_w, \qquad \lambda_x \doteq 0. \tag{34}$$

To make the convergence property applicable to state feedback nonlinear systems, the only change needed is to redefine

$$F(\mu;k) \doteq \lambda_e \|\mu\|_{\{k-r,\dots,k-1\}}. \tag{35}$$

A sufficient condition for the controller to have the convergence property remains $\lambda_e/N < 1$.

The above discussion leads to our third stability result:

*Theorem 3:* Consider a state feedback nonlinear system:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{w}(t)), \qquad \boldsymbol{z}_k = Q(\boldsymbol{x}_k;c_k,\mu_k) \tag{36}$$

where $f$ has the Lipschitz property (33), and for which there exists a static feedback $\boldsymbol{u} = k(\boldsymbol{x})$ which renders the dynamics $\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}, k(\boldsymbol{x}+\boldsymbol{e}), \boldsymbol{w})$ ISS with respect to $\boldsymbol{e}$ and $\boldsymbol{w}$. If $e^{T_s L_x}/N < 1$ then there exists a choice of $\alpha$ and $P$ with which the controller has the convergence property with $F(\mu;k)$ defined in (35). With this choice of $\alpha$ and $P$ and a choice of $\Omega_{out} > e^{T_s L_x}$ and $s > 0$, the aggregate state of the system will satisfy the parameterized ISS property if it can be guaranteed that $\|\boldsymbol{x}\| < l_x$ and $\|\boldsymbol{w}\| < l_w$. This indeed can be guaranteed for $|\boldsymbol{x}(0)| < x_{max}$ and $\|\boldsymbol{w}\| < w_{max}$ such that

$$\beta(x_{max},0;s) + \gamma(w_{max};s) \le l_x \qquad \text{and} \qquad w_{max} \le l_w \tag{37}$$

where $\beta$ and $\gamma$ come from (19). Therefore the aggregate state satisfies the local parameterized ISS property. If the Lipschitz property holds globally, then the aggregate state satisfies the parameterized ISS property.

A natural question would be what is the necessary number of quantizations regions needed to achieve ISS for a given bound on $|\boldsymbol{x}(0)|$ and $\|\boldsymbol{w}\|$. Unfortunately, the theorem does not give a direct answer to this question. Nevertheless, we can say the following: Given $x_{max}$, $l_w = w_{max}$, $l_x$, $L_x$ and $L_w$ such that (33) holds, and $\lambda_e = e^{T_s L_x}$, if

$$\beta_x(x_{max},0) + \gamma_{x,w}(w_{max}) + \gamma_{x,e}\left(\max\left\{\lambda_e\left(x_{max} + \frac{w_{max}}{\lambda_e - 1}\right), \frac{\lambda_e^3}{\lambda_e - 1} w_{max}\right\}\right) < l_x$$

holds, where $\beta_x$, $\gamma_{x,e}$ and $\gamma_{x,w}$ are the ISS gains of the state feedback control law, then there exist appropriate design parameters $P$, $\Omega_{out}$, $\alpha$, $N$ and $s$ with which the closed-loop system will have the local parameterized ISS property. In this way we can reach a semi-global result very similar to the one recently proved in [31], although that paper follows a somewhat different approach and also allows modeling errors and measurement disturbances.

The proof of Theorem 3 follows the same lines as the proof of Theorem 1 and it is therefore omitted. See also [11] for a similar result but without disturbances.

## VII. Conclusions

In this paper we showed how to achieve input-to-state stability with respect to external disturbances using measurements from a dynamic quantizer. We showed that our technique is applicable to output feedback, is robust to modeling errors, and can work with data rates arbitrarily close to the minimum data rate for unperturbed systems. We also showed that our technique can be extended to nonlinear systems.

The following are some problems which were raised by this work, and should be addressed in future research. In the state feedback case, we show what is the necessary and sufficient number of quantization regions required for Input-to-State Stability. In the output feedback case, however, we can only show that a given number of quantization regions is sufficient based on which observer is implemented. It is possible that using another observer a smaller number of quantization regions will be sufficient. This raises the question of what is the optimal observer. When addressing this question one usually needs to consider also the computational resources that are available for the observer.

In the recent paper [32], the method presented here was extended to systems with (possibly time varying) delays in addition to state quantization. Although external disturbances were not considered, we did rely on the ISS property established here after we showed that error signals which arise due to delays can be regarded as external disturbances.

Our analysis only considers the worst-case scenario, defined by the bound on the magnitude of the actual disturbance. In many applications the disturbance can be modeled to follow a certain distribution which rarely produces the worst-case disturbance. By utilizing the knowledge of the underlying distribution, it might be possible to get a more accurate description of the behavior of the system. It should also be possible in this case to provide better tools for choosing the design parameters under different performance requirements.

## Appendix A: Proofs of the Technical Lemmas

*Proof of Lemma 1:* Assume $\alpha$ satisfies $\sigma_{pi} + \frac{\alpha}{N} \le 1$ and for simplicity assume also that $P$ is a multiple of $r$. Then for all $l \in \{1,\dots,P/r-1\}$:

$$\|\mu'\|_{lr,\dots,(l+1)r-1} \le \sigma_{pi}^l + \sum_{m=0}^{l-1} \sigma_{pi}^m \frac{\alpha}{N} \doteq V(l).$$

Because $\sigma_{pi} < 1$ we have that $V(l)$ converges to $\frac{\alpha}{N(1-\sigma)}$ as $l \to \infty$. We also have

$$\|\mu'\|_{P-r,\dots,P-1} \le \max\left\{\frac{N}{N-2}\sigma_{pi} V(P/r-1) + \frac{\alpha}{N-2}, \left(\frac{N}{N-2}\sigma_{pi}\right)^r V(P/r-1) + \sum_{m=0}^{r-1}\left(\frac{N}{N-2}\sigma_{pi}\right)^m \frac{\alpha}{N-2}\right\}.$$

Since we can make $V(P/r-1)$ arbitrarily small by taking $P$ to be large enough and $\alpha$ to be small enough, we can make $\|\mu'\|_{P-r,\ldots,P-1}<1$, which satisfies the convergence property. ■

We will use the following definition in the proofs below:

$$\widetilde{D} \doteq \begin{bmatrix} -C & -CA_d^{-1} & \cdots & -CA_d^{-r+2} \\ 0 & -C & \cdots & -CA_d^{-r+3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -C \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

*Proof of Lemma 3:* Set $\lambda=\|\mu\|_{\{k'-r+1,\ldots,k'\}}$. Between time steps $k'+1$ and $k'+P$, $\mu$ is updated according to (13) or (14). Note that $F(\mu;k)$ depends linearly, with positive coefficients, on $\mu_{k-r},\ldots,\mu_{k-1}$. Therefore, it is easy to see by induction from $k=k'+1$ to $k=k'+P$ that $\mu_k\leq\lambda\mu'_{k-k'+r-1}$. As we have that condition (16) holds, the result of the lemma follows. ■

*Proof of Lemma 4:* The settings $mode(k'+1)=update$ and $p=0$ imply that for $m\in\{k'-r+1,\ldots,k'\}$ we had $saturated(m)=$ **false** and either $mode(m)=capture$ or $mode(m)=detect$. The structure of our quantizer is such that if $saturated(m)=$ **false** for some $m$, then $|\tilde{\boldsymbol{y}}_m|<\mu_m$ where $\tilde{\boldsymbol{y}}_m\doteq\boldsymbol{z}_m-\boldsymbol{y}_m$ denotes the quantization error. The observations can be written as

$$\begin{aligned}\boldsymbol{z}_{k-l} =& CA_d^l\boldsymbol{x}_k - C\sum_{i=1}^{l}A_d^{-i}\boldsymbol{u}_{k-l+i-1}^d - C\sum_{i=1}^{l}A_d^{-i}\boldsymbol{w}_{k-l+i-1}^d \\ &+\tilde{\boldsymbol{y}}_{k-l}.\end{aligned} \tag{38}$$

Since the state estimate (10) was chosen so that $\hat{\boldsymbol{x}}_k=\boldsymbol{x}_k$ in the absence of measurement errors and disturbances, we get together with (38) that

$$\tilde{\boldsymbol{x}}_k^+ = G\begin{bmatrix}\tilde{\boldsymbol{y}}_{k-r+1}\\ \vdots \\ \tilde{\boldsymbol{y}}_k\end{bmatrix} + G\widetilde{D}\begin{bmatrix}\boldsymbol{w}_{k-r+1}^d\\ \vdots \\ \boldsymbol{w}_{k-1}^d\end{bmatrix}. \tag{39}$$

When taking the next measurement at time step $k+1$, the distance between the real output, $\boldsymbol{y}_{k+1}$, and the center of the quantizer $C\hat{\boldsymbol{x}}_{k+1}^-$ is

$$\begin{aligned}\left|\boldsymbol{y}_{k+1}-C\hat{\boldsymbol{x}}_{k+1}^-\right| &= \left|CA_d\tilde{\boldsymbol{x}}_k^+ + C\boldsymbol{w}_k^d\right| \leq \\ &F(\mu;k+1)+\left\|\left[CA_dG\widetilde{D}\,|\,C\right]\right\|\left\|\boldsymbol{w}^d\right\|_{[k'-r+1,k]}.\end{aligned} \tag{40}$$

Given that (20) holds with

$$\zeta_D \doteq \left\|\left[CA_dG\widetilde{D}\,|\,C\right]\right\|,$$

we have from (13) that

$$\left|\boldsymbol{y}_{k+1}-C\hat{\boldsymbol{x}}_{k+1}^-\right|\leq N\mu_{k+1}. \tag{41}$$

The structure of our quantizer guarantees in this case that $\left|\tilde{\boldsymbol{y}}_{k+1}\right|\leq\mu_{k+1}$. We can now repeat these arguments and show that (39)–(41) holds for all $k\in\{k',\ldots,k'+P-r\}$.

At time steps $k'+P-r$ the controller will switch to $mode(k'+P-r+1)=detect$, and we will have for $l=P-r+1$ that $\left|\boldsymbol{y}_{k'+l}-C\hat{\boldsymbol{x}}_{k'+l}^-\right|\leq(N-2)\,\mu_{k'+l}$. This guarantees that both $\left|\tilde{\boldsymbol{y}}_{k'+l}\right|\leq\mu_{k'+l}$ and $saturated(k'+l)=$ **false**, thus $mode(k'+l+1)=detect$. Again, we can repeat these arguments for $l\in\{P-r+2,\ldots,P\}$ with the exception that for $l=P$ the controller will set $mode(k'+l+1)=update$.

Based on (39) we can bound the estimation error for $l\in\{0,\ldots,P-1\}$ as

$$\begin{aligned}\left|\tilde{\boldsymbol{x}}_{k'+l}^+\right| \leq& \|G\|\,\|\mu\|_{\{k'-r+1,\ldots,k'+l\}} + \\ &\left\|G\widetilde{D}\right\|\left\|\boldsymbol{w}^d\right\|_{\{k'-r+1,\ldots,k'+l-1\}}\\ \leq&\zeta_\mu\,\|\mu\|_{\{k'-r+1,\ldots,k'\}}\end{aligned}$$

where

$$\zeta_\mu \doteq \|G\|\,\|\mu'\|_{\{0,\ldots,r+P-2\}} + \left\|G\widetilde{D}\right\|\frac{\alpha}{\zeta_D}.$$

Note that in the definition of $\zeta_\mu$ we used the constants $\mu'$'s defined in (15). ■

*Proof of Lemma 5:* If (20) does not hold, then it will not necessarily be true that $\|\tilde{\boldsymbol{y}}_k\|\leq\mu_k$, $\forall k\in\{k'+1,\ldots,k'+P-r\}$. However, since now we have that

$$\begin{aligned}\|\tilde{\boldsymbol{y}}\|_{\{k'-r+1,\ldots,k'\}} &\leq \|\mu\|_{\{k'-r+1,\ldots,k'\}}\\ &\leq\frac{1}{\alpha}\zeta_D\left\|\boldsymbol{w}^d\right\|_{\{k'-r+1,k'+P\}}\end{aligned} \tag{42}$$

we can still bound the estimation error as follows. For $k\in\{k',\ldots,k_3\}$ we have

$$\begin{aligned}\left\|\tilde{\boldsymbol{x}}_k^+\right\| \leq& \|G\|\,\|\tilde{\boldsymbol{y}}\|_{\{k-r+1,\ldots,k\}} + \\ &\left\|G\widetilde{D}\right\|\left\|\boldsymbol{w}^d\right\|_{\{k-r+1,\ldots,k-1\}}\\ \left\|\tilde{\boldsymbol{y}}_{k+1}\right\| \leq& \|CA_d\|\left\|\tilde{\boldsymbol{x}}_k^+\right\| + \|C\|\left|\boldsymbol{w}_k^d\right|.\end{aligned}$$

Iterating these two inequalities and combining with (42) we get $\left|\tilde{\boldsymbol{x}}_k^+\right|\leq\zeta_w\left\|\boldsymbol{w}^d\right\|_{k'-r+1,\ldots,k-1}$ where

$$\begin{aligned}\zeta_w \doteq& \|G\|\,\|CA_dG\|^P\frac{1}{\alpha}\zeta_D + \left\|G\widetilde{D}\right\| + \\ &\sum_{m=1}^{P}\|G\|\,\|CA_dG\|^{m-1}\left(\left\|CA_dG\widetilde{D}\right\|+\|C\|\right).\end{aligned}$$

■

*Proof of Lemma 6:* Let $k_1$ be the first time step after $k_0$ such that $mode(k_1+1)=update$ (let $k_1=\infty$ if no such time step exists). We now have that for all $l\in\{0,\ldots,k_1-k_0\}$

$$\begin{aligned}\left|\tilde{\boldsymbol{x}}_{k_0+l}^-\right| &\leq \|A_d\|^l\left|\tilde{\boldsymbol{x}}_{k_0}\right| + \sum_{m=0}^{l-1}\|A_d\|^{l-m-1}\left|\boldsymbol{w}_{k_0+m}^d\right|\\ &\leq\|A_d\|^l\left|\tilde{\boldsymbol{x}}_{k_0}\right| + \frac{\|A_d\|^l-1}{\|A_d\|-1}\left\|\boldsymbol{w}^d\right\|\\ &\leq\|A_d\|^l\left(\left|\tilde{\boldsymbol{x}}_{k_0}\right|+\zeta_C\left\|\boldsymbol{w}^d\right\|\right)\end{aligned} \tag{43}$$

where $\zeta_C\doteq\frac{1}{\|A_d\|-1}$. Now, the zoom factor grows as $\mu_{k_0+l}=\mu_{k_0}\Omega_{out}^l$. Define

$$T_1^*(\nu;\rho)\doteq\max\left\{0,\log_{\Omega_{out}/\|A_d\|}\left(\frac{\|C\|\,\nu}{\rho\,(N-2)}\right)+1\right\}+r-1$$

and note that when $\rho$ is fixed, $T_1^*(\cdot;\rho)$ is a nondecreasing function. Assuming $mode(k)=capture$ $\forall k\in$

$\left\{k_0+1,\ldots,k_0+\left\lfloor T_1^*\left(|\boldsymbol{x}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)\right\rfloor\right\}$, we will have

$$\left|\boldsymbol{y}_{k_0+\lfloor T_1^*\rfloor-r+1}-C\hat{\boldsymbol{x}}^-_{k_0+\lfloor T_1^*\rfloor-r+1}\right|\le\|C\|\left|\tilde{\boldsymbol{x}}^-_{k_0+\lfloor T_1^*\rfloor-r+1}\right| \le(N-2)\,\mu_{k_0+\lfloor T_1^*\rfloor-r+1}.$$

Thus $saturated(k_0+\lfloor T_1^*\rfloor-r+1)=$ **false** as well as $saturated(k_0+\lfloor T_1^*\rfloor+l)=$ **false** for $l=-r+2,\ldots,0$ which guarantees that $k_1\le k_0+T_1^*<\infty$ where $k_1$ is the first time step after $k_0$ such that $mode\,(k_1+1)=update$ and $p_{k_1}=0$. Using (43) we can bound the estimation error until the controller switches to the *measurement update* mode at $k_1$ by

$$\begin{aligned}\|\tilde{\boldsymbol{x}}\|_{\{k_0,\ldots,k_1\}}\le&\tilde{\delta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)\\ \tilde{\delta}_1(\nu;\rho)\doteq&\|A_d\|^{T_1^*(\nu;\rho)}\nu.\end{aligned}$$

Note also that $\tilde{\delta}_1\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)\le\mu_{k_0}\zeta_b\Omega_{out}^{T_1^*\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\|\boldsymbol{w}^d\|;\mu_{k_0}\right)}$ where $\zeta_b\doteq\frac{(N-2)}{\|C\|}$. ■

*Proof of Lemma 7:* Assume first that $mode(k_0)=capture$ and consider the case $|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|\le\frac{\mu_{k_0}}{\|C\|}$. Following the same arguments as in Lemma 6 which led to (43), we can write for $l\in\{1,\ldots,r\}$:

$$\begin{aligned}\|C\|\,|\tilde{\boldsymbol{x}}_{k_0+l}|\le&\|C\|\,\|A_d\|^l\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|\right)\\ \le&\mu_{k_0}\Omega_{out}^l=\mu_{k_0+l}.\end{aligned}$$

This implies that if $mode(k_0)=capture$, then at a sampling time not later than $k_0+r$ the controller will switch to the *measurement update* mode. If for some time step $k$ the following holds

$$\left|\boldsymbol{y}_k-C\hat{\boldsymbol{x}}_k^-\right|\le\|C\|\left|\tilde{\boldsymbol{x}}_k^-\right|\le\mu_k \tag{44}$$

then the output from the quantizer will be such that $\boldsymbol{z}_k=\boldsymbol{c}=C\hat{\boldsymbol{x}}_k^-$. If for some time step $k'$ (44) is true $\forall k\in\{k'-p,\ldots,k'\}$, and $\hat{\boldsymbol{x}}_{k'}^+$ is updated with $G(\boldsymbol{z};\boldsymbol{u}^d;k')$, then we will have $\hat{\boldsymbol{x}}_{k'}^+=\hat{\boldsymbol{x}}_{k'}^-$. This implies that $\tilde{\boldsymbol{x}}_{k'}=A_d\tilde{\boldsymbol{x}}_{k'-1}+\boldsymbol{w}_{k'-1}^d$. In turn, this means that as long as (44) holds $\forall l\in\{0,\ldots,k^*-k_0\}$, even if $mode(k^*)\neq capture$, then so does (43) $\forall l\in\{0,\ldots,k^*-k_0\}$.

Now define

$$\begin{aligned}\xi(\nu;\rho)\doteq&\left(\frac{1}{\rho\varsigma}\right)^{\frac{\log\left(\|A_d\|^P\right)}{\log(\sigma)-\log\left(\|A_d\|^P\right)}}\left(\|C\|\,\nu\right)^{\frac{\log(\sigma)}{\log(\sigma)-\log\left(\|A_d\|^P\right)}}\\ T_2^*(\nu;\rho)\doteq&P\left\lfloor\log_\sigma\left(\frac{\xi(\nu;\rho)}{\rho\varsigma}\right)\right\rfloor\\ \varsigma\doteq&\min_{k\in\{r,\ldots,r+P-1\}}\mu'(k)\le\sigma.\end{aligned}$$

Note that in the definition of $\varsigma$ we use the $\mu'$'s defined in (15) and we assume without loss of generality that $\varsigma>0$. Assume also that $|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|$ is sufficiently small such that $T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)\ge r+P$. We defined $\xi$ and $T_2^*$ such that we will have for all $k\in\{k_0,\ldots,k_0+T_2^*\left(\|\boldsymbol{w}\|\right)\}$

$$\begin{aligned}\mu_k\ge&\mu_{k_0}\varsigma\sigma^{T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\|\boldsymbol{w}^d\|;\mu_{k_0}\right)/P}\\ >&\xi\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)\end{aligned} \tag{45}$$

and

$$\begin{aligned}&\|C\|\,\|\tilde{\boldsymbol{x}}\|_{\left\{k_0,\ldots,k_0+T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\|\boldsymbol{w}^d\|;\mu_{k_0}\right)\right\}}\\ &\le\|A_d\|^{T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\|\boldsymbol{w}^d\|;\mu_{k_0}\right)}\|C\|\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|\right)\\ &\le\left(\frac{\xi\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)}{\mu_{k_0}\varsigma}\right)^{\frac{\log\left(\|A_d\|^P\right)}{\log(\sigma)}}\|C\|\times\\ &\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|\right)=\xi\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right).\end{aligned} \tag{46}$$

In deriving the first inequality in (46) we used (43) to bound the estimation error – even though it is not true that $mode(k_0+l)=capture\ \forall l<T_2^*$, we can still use (43) since (45) and (46) imply that (44) holds. The proof is completed by setting

$$\tilde{\delta}_2\left(\nu;\rho\right)\doteq\frac{\xi\left(\nu;\rho\right)}{\|C\|},\qquad\zeta_s\doteq\Omega_{out}^r/\varsigma \tag{47}$$

and letting $\varepsilon\left(\cdot\right)>0$ be any class $\mathcal{K}$ function such that $\varepsilon\left(\rho\right)\le\frac{\rho}{\|C\|}$ and $T_2^*\left(\varepsilon\left(\rho\right);\rho\right)\ge r+P$. Note that the function $\tilde{\delta}_2\left(\cdot;\rho\right)$ is a class $\mathcal{K}_\infty$ function for each fixed $\rho$, and that $\tilde{\delta}_2\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\left\|\boldsymbol{w}^d\right\|;\mu_{k_0}\right)\le\mu_{k_0}\zeta_s\sigma^{T_2^*\left(|\tilde{\boldsymbol{x}}_{k_0}|+\zeta_C\|\boldsymbol{w}^d\|;\mu_{k_0}\right)/P}/\,\|C\|$. ■

## Appendix B: Small-Gain Theorem for Local Practical Parameterized ISS

The following is a modification of the small-gain theorem ([23, Theorem 2.1]). It states that the interconnection of an ISS system and a parameterized ISS system, under a small-gain condition, results in a local practical ISS system. The modification is due to the additional third signal $\mu$, that does not have an ISS relation with respect to the disturbance, and due to the fact that the small-gain condition only holds locally. We believe the results of this appendix have independent interest beyond the scope of proving Theorem 2. Indeed, they had already been used (in a different context) in [32].

*Theorem 4:* Consider two systems whose state variables, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, satisfy the ISS and the parameterized ISS properties, respectively:

$$\begin{aligned}|\boldsymbol{x}_1\left(t\right)|\le&\beta_1\left(|\boldsymbol{x}_1\left(t_0\right)|,t-t_0\right)+\gamma_1\left(\|\boldsymbol{x}_2\|_{[t_0,t]}\right)+\\ &\gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\\ |\boldsymbol{x}_2\left(t\right)|\le&\beta_2\left(|\boldsymbol{x}_2\left(t_0\right)|,t-t_0;\mu\left(t_0\right)\right)+\\ &\gamma_2\left(\delta\,\|\boldsymbol{x}_1\|_{[t_0,t]};\mu\left(t_0\right)\right)+\gamma\left(\|\boldsymbol{w}\|_{[t_0,t]};\mu\left(t_0\right)\right)\\ \mu\left(t\right)\le&\gamma_\mu\left(\|\boldsymbol{x}_2\|_{[t_0,t]},\mu\left(t_0\right)\right),\qquad\forall t\ge t_0\ge0.\end{aligned} \tag{48}$$

Assume the first trajectory, $\boldsymbol{x}_1$, is continuous. Then the interconnected system will satisfy the local practical input-to-state stability property: $\exists\delta_{\max},x_{\max},w_{\max}\in\mathbb{R}_{>0}$ such that $\forall\delta\le\delta_{\max}$, $\forall\,|\boldsymbol{x}_1\left(0\right)|<x_{\max}$, $\forall\,|\boldsymbol{x}_2\left(0\right)|<x_{\max}$, $\forall\,\|\boldsymbol{w}\|_{[0,t]}<w_{\max}$,

$$\begin{aligned}|\boldsymbol{x}_1\left(t\right)|\le&\beta_{ic}\left(\left|\begin{pmatrix}\boldsymbol{x}_1\left(t_0\right)\\ \boldsymbol{x}_2\left(t_0\right)\end{pmatrix}\right|,t-t_0\right)+\gamma_{ic}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)+\\ &\lambda\left(\delta\right)\end{aligned} \tag{49}$$

for all $t \geq t_0 \geq 0$ where $\beta_{ic}$ is of class $\mathcal{KL}$ and $\gamma_{ic}$ and $\lambda$ are of class $\mathcal{K}_\infty$.

The proof of Theorem 4 will come after the following intermediate results.

*Lemma 8:* Assume for some $t_0$ a signal $\boldsymbol{x}$ satisfies

$$|\boldsymbol{x}(t)| \leq d + \gamma\left(\|\boldsymbol{x}\|_{[t_0,t]}\right), \qquad \forall t \geq t_0, \tag{50}$$

and

$$\lim_{\tau \nearrow t} |\boldsymbol{x}(\tau)| \geq |\boldsymbol{x}(t)|, \qquad \forall t \geq t_0 \tag{51}$$

(any discontinuity in the signal results in a decrease of the norm of the signal). Assume further that for some $r_2 > r_1 > 0$, $\lambda < 1$

$$\gamma(r) < \lambda r \qquad \forall r \in [r_1, r_2] \tag{52}$$

and

$$x_{\max} \doteq \max\left\{r_1, \frac{1}{1-\lambda} d\right\} < r_2. \tag{53}$$

Then given that $|\boldsymbol{x}(t_0)| \leq x_{\max}$, we have

$$\|\boldsymbol{x}\|_{[t_0,\infty)} \leq x_{\max}.$$

Note that in particular every continuous signal satisfies (51). This lemma is stated more generally than what is needed to prove Theorem 4. In the proof of Theorem 4 it is applied on the state $\boldsymbol{x}_1$ which is assumed to be continuous. When Theorem 4 is used to prove Theorem 2, $\boldsymbol{x}_1$ corresponds to the state of the system, which is indeed continuous. The state $\boldsymbol{x}_2$, on the other hand, corresponds to the estimation error which may be discontinuous at sampling times. We note that in the state feedback case the estimation error does satisfy (51) due to the construction of our quantizer, and so we could have applied Lemma 8 on $\boldsymbol{x}_2$ instead of on $\boldsymbol{x}_1$. This observation will be useful when considering other extensions such as delays [32].

*Proof of Lemma 8:* Assume on the contrary that there exists $t' \geq t_0$ such that $|\boldsymbol{x}(t')| > x_{\max}$. Choose $\varepsilon = x_{\max} + \min\left\{\|\boldsymbol{x}\|_{[t_0,\infty)} - x_{\max}, r_2 - x_{\max}\right\}/2$ so that $t = \inf\left\{\tau \geq t_0 \middle| |x(\tau)| \geq \varepsilon\right\}$ is well-defined. By definition of $t$, $\|\boldsymbol{x}\|_{[t_0,t)} \leq x_{\max} + \varepsilon$ and from $|\boldsymbol{x}(t_0)| \leq x_{\max}$ and (51), $t > t_0$ and $|\boldsymbol{x}(t)| = x_{\max} + \varepsilon$. Thus $\|\boldsymbol{x}\|_{[t_0,t]} = x_{\max} + \varepsilon < r_2$. From (50) and (52) we can now write $\|\boldsymbol{x}\|_{[t_0,t]} \leq d + \lambda \|\boldsymbol{x}\|_{[t_0,t]}$, and conclude using (53) that $\|\boldsymbol{x}\|_{[t_0,t]} \leq x_{\max}$. This contradicts $\|\boldsymbol{x}\|_{[t_0,t]} = x_{\max} + \varepsilon$. ■

A corollary of the small-gain theorem [23, Theorem 2.1] gives us the following local result:

*Lemma 9:* Given $\beta_1, \beta_2 \in \mathcal{KL}$, $\gamma_{1,x}, \gamma_{2,x} \in \mathcal{K}_\infty$, and $\rho < 1$, there exists $\beta \in \mathcal{KL}$ and $\gamma, \lambda_1, \lambda_2, \lambda_0 \in \mathcal{K}_\infty$ such that for every $r_1 > r_0 > 0$ which satisfy the small-gain condition

$$\gamma_{1,x}(\gamma_{2,x}(r)) \leq \rho r, \qquad \forall r \in [r_0, r_1],$$

the following property holds. For every three signals $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{w}$ satisfying $\forall t \geq t_0 \geq 0$

$$\begin{aligned}|\boldsymbol{x}_1(t)| \leq& \beta_1(|\boldsymbol{x}_1(t_0)|, t-t_0) + \gamma_{1,x}\left(\|\boldsymbol{x}_2\|_{[t_0,t]}\right) + \\ & \gamma_{1,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + d_1 \\ |\boldsymbol{x}_2(t)| \leq& \beta_2(|\boldsymbol{x}_2(t_0)|, t-t_0) + \gamma_{2,x}\left(\|\boldsymbol{x}_1\|_{[t_0,t]}\right) + \\ & \gamma_{2,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) + d_2\end{aligned}$$

for some $\gamma_{1,w}, \gamma_{2,w} \in \mathcal{K}$ and $d_1, d_2 \in \mathbb{R}_{\geq 0}$, if it can be guaranteed that $\|\boldsymbol{x}_1\|_{[0,\infty]} \leq r_1$, then $\forall t \geq t_0 \geq 0$:

$$\begin{aligned}|\boldsymbol{x}_1(t)| \leq& \beta\left(\left|\begin{array}{c}\boldsymbol{x}_1(t_0)\\ \boldsymbol{x}_2(t_0)\end{array}\right|, t-t_0\right) + \gamma\left(\gamma_{1,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\right) + \\ & \gamma\left(\gamma_{2,w}\left(\|\boldsymbol{w}\|_{[t_0,t]}\right)\right) + \lambda_1(d_1) + \lambda_2(d_2) + \lambda_0(r_0).\end{aligned}$$

*Proof of Theorem 4:* Choose arbitrary $r_1 > r_0 > 0$, $\delta'_{\max} > 0$, $\bar{\mu}$ such that $\bar{\mu} > \gamma_\mu(\gamma_2(\delta'_{\max} r_1; \mu(0)); \mu(0))$, $\rho < 1$ and consider the following small-gain condition:

$$\gamma_1(\gamma_2(\delta r, \mu)) \leq \rho r, \qquad \forall r \in [r_0, r_1] \subset \mathbb{R}_{\geq 0}, \quad \forall \mu \in [0, \bar{\mu}]. \tag{54}$$

For every fixed $\mu$, $\gamma_1(\cdot)$ and $\gamma_2(\cdot;\mu)$ are of class $\mathcal{K}_\infty$. Thus for every $r \in [r_0, r_1]$ and every $\mu \in [0, \bar{\mu}]$ there exists a small enough but strictly positive $\delta(r,\mu)$ for which the small-gain condition holds. Set $\delta_{\max} \doteq \min\left\{\delta'_{\max}, \min_{\substack{r \in [r_0, r_1] \\ \mu \in [0,\bar{\mu}]}} \delta(r,\mu)\right\} > 0$.

Since $\rho$ in (54) is strictly smaller than 1, there exist $\alpha > 0$ and $\rho' < 1$ such that

$$\gamma_1((1+\alpha)\gamma_2(\delta_{\max} r; \mu)) \leq \rho' r, \ \forall r \in [r_0, r_1], \ \forall \mu \in [0, \bar{\mu}]. \tag{55}$$

For all nondecreasing functions $\gamma$ and all $\alpha > 0$, $a > 0$ and $b > 0$, we have $\gamma(a+b) \leq \gamma((1+\alpha)a) + \gamma((1+1/\alpha)b)$. Using this and (55) we can derive $\forall t \geq 0$,

$$\begin{aligned}|\boldsymbol{x}_1(t)| \leq& \beta_1(|\boldsymbol{x}_1(0)|, 0) + \gamma(\|\boldsymbol{w}\|) + \\ & \gamma_1\Bigg(\beta_2(|\boldsymbol{x}_2(0)|, 0; \mu(0)) + \gamma_2\left(\delta_{\max}\|\boldsymbol{x}_1\|_{[0,t]}\right) + \\ & \gamma(\|\boldsymbol{w}\|; \mu(0))\Bigg) \\ \leq& \beta_1(|\boldsymbol{x}_1(0)|, 0) + \\ & \gamma_1\left((1+\alpha)\gamma_2\left(\delta_{\max}\|\boldsymbol{x}_1\|_{[0,t]}; \mu(0)\right)\right) + \gamma(\|\boldsymbol{w}\|) \\ & \gamma_1\Bigg(\left(1+\frac{1}{\alpha}\right) \times \\ & (\beta_2(|\boldsymbol{x}_2(0)|, 0; \mu(0)) + \gamma(\|\boldsymbol{w}\|; \mu(0)))\Bigg).\end{aligned}$$

Define

$$\begin{aligned}& s_\infty(|\boldsymbol{x}_1(0)|, |\boldsymbol{x}_2(0)|, \mu(0), \|\boldsymbol{w}\|) \doteq \\ & \frac{1}{1-\rho'}\Bigg((\beta_1(|\boldsymbol{x}_1(0)|, 0) + \gamma(\|\boldsymbol{w}\|)) + \\ & \gamma_1\left(\left(1+\frac{1}{\alpha}\right)(\beta_2(|\boldsymbol{x}_2(0)|, 0; \mu(0)) + \gamma(\|\boldsymbol{w}\|; \mu(0)))\right)\Bigg).\end{aligned}$$

By the choice of $\bar{\mu}$, it is always possible to find $s_{\max} < r_1$, $x_{\max} > 0$, $w_{\max} > 0$ such that

$$\begin{aligned}& s_\infty(|\boldsymbol{x}_1(0)|, |\boldsymbol{x}_2(0)|, \mu(0), \|\boldsymbol{w}\|) \leq s_{\max} \leq r_1, \\ & \gamma_\mu\big(\beta_2(|\boldsymbol{x}_2(0)|, 0; \mu(0)) + \\ & \quad \gamma_2(\delta_{\max} s_{max}; \mu(0)) + \gamma(\|\boldsymbol{w}\|; \mu(0)), \mu(0)\big) < \bar{\mu}\end{aligned}$$

$$\forall \left|\boldsymbol{x}_1(0)\right| < x_{\max},\ \forall \left|\boldsymbol{x}_2(0)\right| < x_{\max},\ \forall \|\boldsymbol{w}\|_{[0,t]} < w_{\max}. \tag{56}$$

Given that (56) holds, we can use Lemma 8 to get $\|\boldsymbol{x}_1\|_{[0,\infty))]} \leq s_{\max}$. Using (48) we can also derive the bound

$$\|\mu\| \leq \gamma_\mu\big(\beta_2\left(\left|\boldsymbol{x}_2(0)\right|, 0; \mu(0)\right) + \gamma_2\left(\delta_{\max} s_{max}; \mu(0)\right) + \gamma\left(\|\boldsymbol{w}\|; \mu(0)\right), \mu(0)\big) < \bar{\mu}.$$

And with this, we can write

$$\begin{aligned}
\left|\boldsymbol{x}_1(t)\right| \leq& \beta_1\left(\left|\boldsymbol{x}_1(t_0)\right|, t-t_0\right) + \gamma_1\left(\|\boldsymbol{x}_2\|_{[t_0,t]}\right) + \\
&\gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}\right) \\
\left|\boldsymbol{x}_2(t)\right| \leq& \max_{\mu\in[0,\bar{\mu}]} \beta_2\left(\left|\boldsymbol{x}_2(t_0)\right|, t-t_0; \mu\right) + \\
&\max_{\mu\in[0,\bar{\mu}]} \gamma_2\left(\delta_{\max}\|\boldsymbol{x}_1\|_{[t_0,t]}; \mu\right) + \\
&\max_{\mu\in[0,\bar{\mu}]} \gamma\left(\|\boldsymbol{w}\|_{[t_0,t]}; \mu\right)
\end{aligned}$$

for all $t \geq t_0 \geq 0$. Note that for every fixed $\mu \in \mathbb{R}_{\geq 0}$ the function $\beta_2(\cdot,\cdot;\mu)$ is a function of class $\mathcal{KL}$ and the functions $\gamma_2(\cdot;\mu)$ and $\gamma(\cdot;\mu)$ are of class $\mathcal{K}_\infty$. They are also all continuous in $\mu$. Thus taking the maximum of these functions over $\mu$ is well defined and does not change their $\mathcal{KL}$/$\mathcal{K}_\infty$ characteristics. Note that we can actually satisfy the following small-gain condition $\forall\delta \leq \delta_{\max}$:

$$\gamma_1\left(\gamma_2\left(\delta r, \mu\right)\right) \leq \rho r,\ \forall r \in \left[\lambda(\delta), r_1\right] \subset \mathbb{R}_{\geq 0},\ \forall \mu \in [0, \bar{\mu}].$$

where $\lambda \in \mathcal{K}$. Lemma 9 now gives us (49). ■

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